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LETTER TO THE EDITOR

Double wave description of a single relativistic free electron

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Abstract. This letter introduces two wavefunctions to describe a single relativistic free electron, from which we can obtain the value of an arbitrary physical quantity at any time t. We see that the motion of free electrons is the same as that given by special relativity.

It is well known that in classical Hamiltonian mechanics a set of generalised coordinates alone is not sufficient to describe a single system and a set of generalised momenta must be added simultaneously, i.e. generalised coordinates and generalised momenta together give the complete description of a single system. The situation is the same in quantum mechanics, where a single wavefunction can only describe an ensemble and it is not sufficient to describe a single system. The same conclusion is applicable to its classical limit (Qian and Huang 1986). In a recent article (Huang 1986) we made the suggestion of introducing two wavefunctions jointly to describe a free particle. This suggestion will be called the 'double wave description' (DWD). DWD describes a single system completely and it can be readily generalised to non-relativistic quantum mechanics and relativistic quantum mechanics as a whole. In this letter we shall only use DWD to discuss a relativistic free electron.

Firstly, we consider the relativistic free electron with positive energy $E_+ = (m_0^2c^4 + c^2p^2)^{1/2} = |E|$ and parallel spin. In this case one wavefunction is the de Broglie wave with parameter x_0:

$$\psi_{p_0}^{(1)}(x, t) = \frac{N(p_0)}{\sqrt{2\pi\hbar}} u^{(1)}(p_0) \exp\left(\mathrm{i}\frac{p_0(x - x_0)}{\hbar} - \mathrm{i}\frac{E_+(p_0)t}{\hbar}\right) \tag{1}$$

and another wavefunction is obtained by superposition of all possible positive energy de Broglie waves with equal weight:

$$\phi_{(+)}(x, t) = \int_{-\infty}^{\infty} \mathrm{d}p \sum_{\lambda=1,2} \frac{N(p)}{\sqrt{2\pi\hbar}} u^{(\lambda)}(p) \exp\left(\mathrm{i}\frac{p(x - x_0)}{\hbar} - \mathrm{i}\frac{E_+(p)\hbar}{\hbar}\right) \tag{2}$$

where (Merzbacher 1970)

$$u^{(1)}(p) = \begin{pmatrix} 1 \\ 0 \\ cp/(m_0c^2 + E_+) \\ 0 \end{pmatrix} \qquad u^{(2)}(p) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -cp/(m_0c^2 + E_+) \end{pmatrix} \tag{3}$$

and

$$N(p) = \left(1 + \frac{c^2p^2}{(m_0c^2 + E_+)^2}\right)^{-1/2}. \tag{4}$$

These two waves can be combined to describe the motion of the relativistic free electron. In DWD the physical quantity f measured for a single particle is assumed to be given by

$$\langle f(t)\rangle = \mathrm{Re}\int \phi^{\dagger}_{(+)}(x,t)\hat{f}\psi^{(1)}_{p_0}(x,t)\,\mathrm{d}x. \tag{5}$$

From (5) we obtain the momentum, energy, position and velocity of the electron as follows:

$$\langle p\rangle = \mathrm{Re}\int \phi^{\dagger}_{(+)}(x,t)\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial x}\psi^{(1)}_{p_0}(x,t)\,\mathrm{d}x = p_0 \tag{6}$$

$$\langle E\rangle = \mathrm{Re}\int \phi^{\dagger}_{(+)}(x,t)\mathrm{i}\hbar\frac{\partial}{\partial t}\psi^{(1)}_{p_0}(x,t)\,\mathrm{d}x = E_+(p_0) \tag{7}$$

$$\begin{aligned}\langle x\rangle &= \mathrm{Re}\int \phi^{\dagger}_{(+)}(x,t)x\psi^{(1)}_{p_0}(x,t)\,\mathrm{d}x\\ &= \mathrm{Re}\frac{1}{2\pi\hbar}\sum_{\lambda=1,2}\iint \mathrm{d}p\,\mathrm{d}x\,N(p)N(p_0)u^{(\lambda)\dagger}(p)u^{(1)}(p_0)\\ &\quad\times\exp\frac{\mathrm{i}}{\hbar}\{(p-p_0)x_0+[E_+(p)-E_+(p_0)]t\}\mathrm{i}\hbar\frac{\partial}{\partial p}\exp\left(\frac{-\mathrm{i}}{\hbar}(p-p_0)x\right)\\ &= x_0+\frac{p_0t}{m}\qquad\left(m=\frac{m_0}{\sqrt{1-(v^2/c^2)}}\right)\end{aligned} \tag{8}$$

$$\left\langle\frac{\mathrm{d}x}{\mathrm{d}t}\right\rangle = \mathrm{Re}\int \phi^{\dagger}_{(+)}(x,t)\frac{c^2}{E_+}\hat{p}\psi^{(1)}_{p_0}\,\mathrm{d}x = \frac{c^2}{E_+}p_0 = \frac{p_0}{m}. \tag{9}$$

From (8) we can see that the motion of the electron is the same as that given by special relativity, the initial position is x_0 and the constant velocity is $p_0/m = (p_0/m_0)[1-v^2/c^2)]^{1/2}$. This result cannot be obtained in the usual relativistic quantum mechanics. The ensemble average of (5) gives

$$\bar{f} = \lim_{L\to\infty}\frac{1}{L}\int_{-L/2}^{L/2}\langle f(t)\rangle\,\mathrm{d}x_0 = \lim_{L\to\infty}\frac{2\pi\hbar}{L}\int_{-L/2}^{L/2}\psi^{(1)\dagger}_{p_0}\hat{f}\psi^{(1)}_{p_0}\,\mathrm{d}x \tag{10}$$

which is obtained by the usual method. It should be noted that

$$\begin{aligned}\langle x^2\rangle &= \mathrm{Re}\int \phi^{\dagger}_{(+)}(x,t)x^2\psi^{(1)}_{p_0}(x,t)\,\mathrm{d}x\\ &= \left(x_0+\frac{p_0t}{m}\right)^2 - \sum_{\lambda=1,2}\hbar^2\frac{\partial^2}{\partial p^2}[N(p)N(p_0)u^{(\lambda)\dagger}(p)u^{(\lambda)}(p_0)]\big|_{p=p_0}\\ &= \langle x\rangle^2+\frac{m_0^2\hbar^2c^6}{2E_+^4}.\end{aligned} \tag{11}$$

Equation (11) tells us that the difference between $\sqrt{\langle x^2\rangle}$ and $\langle x\rangle$ approximately equals the Compton wavelength of the electron. For a relativistic free electron with positive energy and antiparallel spin, we have the same result as that for positive energy and parallel spin.

For a relativistic free electron with negative energy $E_- = -|E|$ and parallel spin, two wavefunctions in DWD are

$$\psi^{(3)}_{p_0}(x,t) = \frac{N(p_0)}{\sqrt{2\pi\hbar}} u^{(3)}(p_0) \exp\left(\mathrm{i}\frac{p_0(x-x_0)}{\hbar} - \mathrm{i}\frac{E_-(p_0)t}{\hbar}\right) \tag{12}$$

$$\phi_{(-)}(x,t) = \int_{-\infty}^{\infty} \mathrm{d}p \sum_{\lambda=3,4} \frac{N(p)}{\sqrt{2\pi\hbar}} u^{(\lambda)}(p) \exp\left(\mathrm{i}\frac{p(x-x_0)}{\hbar} - \mathrm{i}\frac{E_-(p)t}{\hbar}\right) \tag{13}$$

where

$$u^{(3)}(p) = \begin{pmatrix} -cp/(mc^2 - E_-) \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad u^{(4)}(p) = \begin{pmatrix} 0 \\ cp/(mc^2 - E_-) \\ 0 \\ 1 \end{pmatrix}. \tag{14}$$

From the assumption of DWD

$$\langle f(t)\rangle = \mathrm{Re} \int \phi^{\dagger}_{(-)}(x,t) \hat{f} \psi^{(3)}_{p_0}(x,t)\,\mathrm{d}x \tag{15}$$

we obtain the physical quantities of the electron as follows:

$$\langle p\rangle = p_0 \tag{16}$$

$$\langle E\rangle = E_-(p_0) = -|E(p_0)| \tag{17}$$

$$\langle x\rangle = x_0 - p_0 t/m \tag{18}$$

$$\left\langle\frac{\mathrm{d}x}{\mathrm{d}t}\right\rangle = -\frac{p_0}{m}. \tag{19}$$

Hence such a particle moves as if it has a negative mass. For a relativistic free electron with negative energy and antiparallel spin, we have the same result as that for negative energy and parallel spin.

Even though for the relativistic free electron we have

$$\langle x^2\rangle - \langle x\rangle^2 = \frac{m_0^2\hbar^2c^6}{2|E|^4} \neq 0 \tag{20}$$

we still have

$$\langle p^2\rangle - \langle p\rangle^2 = 0 \tag{21}$$

and hence the uncertainty relation has no effect for a single relativistic electron. It is a statistical relation which is valid only for an ensemble.

From the above discussion we see that DWD is a deterministic theory which completely describes the motion of a single electron, whereas ordinary quantum mechanics describes an ensemble only. The deterministic character of DWD is consistent with Einstein's stance on quantum mechanics. The statistical nature of ordinary quantum mechanics comes from the incomplete description of a single system by one wavefunction only.

References

Huang X Y 1986 *Phys. Lett.* **115A** 310

Merzbacher E 1970 *Quantum Mechanics* (New York: Wiley) ch 23

Qian S W and Huang X Y 1986 *Phys. Lett.* **115A** 319